STABILITY OF REINFORCED CYLINDRICAL SHELLS IN BENDING BY A MOMENT WITH TORSION

S. V. Astrakharchik and V. V. Kabanov

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The problem of the stability of reinforced circular cylindrical shells in bending and torsion has numerous applications in different regions of modern technology. Most published studies of this problem [1] have been conducted in the classical formulation, with the assumption that the initial stress—strain state is momentless and linear. Below, the problem of the stability of a reinforced cylindrical shell under combination loading is solved on the basis of nonlinearity of the theory in a moment formulation.

We will examine a circular cylindrical shell of length L, thickness h, and radius of curvature r reinforced with rings and made of a material with the modulus of normal elasticity E and Poisson's ratio ν.

We subdivide the shell into m parts lengthwise and k parts about its circumference. The shell is thus represented by a set of m × k finite elements (FE) of natural curvature that are rectangular in plan and k × p (p is the number of rings) curvilinear rod FE reinforcements (the top part of Fig. 1).

We take the following expression [1] for the strains, changes in curvature, and torsion of the FE of the shell:

$$\begin{gathered}\varepsilon_{11} = u_x + 0.5\omega_1^2,\quad \varepsilon_{22} = (1/r)(v_\varphi - w) + 0.5\omega_2^2,\\ \varepsilon_{12} = (1/r)u_\varphi + v_x + \omega_1\omega_2,\ \chi_{11} = \omega_{1x},\ \chi_{22} = (1/r)\omega_{2\varphi},\\ \chi_{12} = \omega_{2x},\ \omega_1 = w_x,\ \omega_2 = (1/r)(w_\varphi + v).\end{gathered} \tag{1}$$

Here, x and φ are the linear and angular coordinates, with the origin at the center of the FE of the shell (Fig. 1); x and φ in the subscripts denote differentiation; u, v, and w denote displacements along the generatrix, the arc, and the normal.

The kinematic relations for the rings [2], written in terms of the displacements of the shell FE, are represented in the form

$$\begin{gathered}\varepsilon = \frac{1}{r}v_\varphi + \frac{1}{R}\left[-w + e_2 w_x + \frac{1}{r}(e_2 u_{\varphi\varphi} - e_1 w_{\varphi\varphi})\right] + \frac{l}{2R^2}w_\varphi^2,\\ \chi_1 = \frac{1}{R^2}[lw - l_1 u_{\varphi\varphi} + l w_{\varphi\varphi} - (l_1 r - l_1 e_1 + l e_2) w_x + (e_1 l_1 - e_2 l) w_{x\varphi\varphi}],\\ \chi_2 = \frac{1}{R^2}(r w_{x\varphi} - u_\varphi),\quad \chi_3 = \frac{1}{R^2}[l_1 w + l u_{\varphi\varphi} + l_1 w_{\varphi\varphi} + (lr - le_1 - l_1 e_2) w_x -\\ - (e_1 l + e_2 l_1) w_{x\varphi\varphi}],\quad l = \cos\alpha,\quad l_1 = \sin\alpha,\quad R = r - e_1,\end{gathered} \tag{2}$$

TABLE 1

m	k	M_t^0/M_s	m	k	M_b^0/M_s	m	k	M_t^0/M_s	m	k	M_b^0/M_s
4	36	2,020	4	40	1,441	12	10	2,361	10	8	1,610
6	36	1,698	6	40	1,243	12	14	1,746	10	16	1,383
10	36	1,568	8	40	1,175	12	16	1,649	10	24	1,241
12	36	1,536	10	40	1,175	12	20	1,536	10	28	1,190
16	36	1.536	12	40	1.175	12	30	1.536	10	32	1.175

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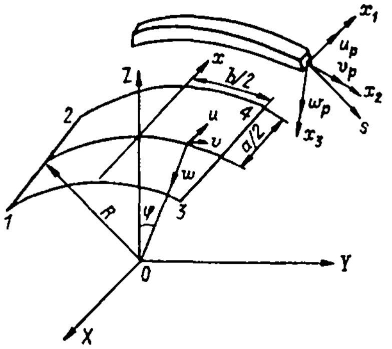

Fig. 1

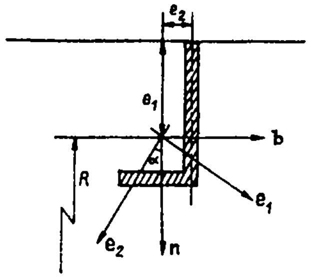

Fig. 2

where e_1 and e_2 are the eccentricities of the location of a ring; α is the angle between the vector of the normal $\mathbf{n}$ to the middle surface of the shell and the unit vector $\mathbf{e}_2$ of the principal central axes of inertia of the cross section of the ring; R is the radius of curvature of the axial line of the ring (Fig. 2).

The static relations for the elements of the shell and reinforcement are written in the form

$$T_{11} = K(\varepsilon_{11} + \nu\varepsilon_{22}),\ T_{22} = K(\varepsilon_{22} + \nu\varepsilon_{11}),\ T_{12} = 0.5K(1 - \nu)\varepsilon_{12},$$
$$M_{11} = D(\chi_{11} + \nu\chi_{22}),\ M_{22} = D(\chi_{22} + \nu\chi_{11}),\ M_{12} = D(1 - \nu)\chi_{12},$$
$$T = EF\varepsilon,\ M_1 = EI_1\chi_1,\ M_3 = EI_2\chi_3,\ M_2 = GI_t\chi_2,$$
$$K = \frac{Eh}{1 - \nu^2},\ D = \frac{Eh^3}{12(1 - \nu^2)},\ G = \frac{E}{2(1 + \nu)},$$

where F is the cross-sectional area of the ring; I_1 and I_2 are the moments of inertia of the cross section of the ring relative to the principal central axes $\mathbf{e}_1$, $\mathbf{e}_2$; GI_t is the stiffness of the ring in torsion.

To consider the displacement of the shell FE as a rigid whole, we take the displacement field, bilinear for u and v and bicubic for w, and we introduce functions which constitute the complete solution of a system of differential equations expressing the triviality of the strains, changes in curvature, and torsion of the shell element (without allowance for the nonlinear terms in Eqs. (1), due to the presumed smallness of the rigid rotation of the FE):

$$u = a_1x\varphi + a_2x + a_3\varphi + a_4 + a_6rs + a_{20}rc,$$
$$v = a_5x\varphi - a_6xc + a_7\varphi + a_8 + a_{20}xs - a_{23}c + a_{24}s,$$
$$w = a_9x^3\varphi^3 + a_{10}x^3\varphi^2 + a_{11}x^3\varphi + a_{12}x^3 + a_{13}x^2\varphi^3 + a_{14}x^2\varphi^2 + a_{15}x^2\varphi + a_{16}x^2 + a_{17}x\varphi^3 + a_{18}x\varphi^2 +$$
$$+ a_{19}x\varphi + a_{20}xc + a_{21}\varphi^3 + a_{22}\varphi^2 + a_{23}s + a_{24}c + a_6xs,$$
$$s = \sin\varphi,\ c = \cos\varphi.$$

The functions describing the rigid displacement of the FE of the shell also cause kinematic relations (2) for the ring element to vanish (without allowance for the nonlinear term in the expression for ε).

As the vector of the generalized nodal displacements, we take $\mathbf{u}_i^T = \{u_1, v_1, w_1, \omega_{11}, \omega_{21}, w_{x\varphi 1}, \ldots, u_4, v_4, w_4, \omega_{14}, \omega_{24}, w_{x\varphi 4}\}$, where the superscript T denotes transposition and the subscripts denote the numbers of the nodes of the shell FE.

We then use the method in [3] to construct nonlinear equations of equilibrium in increments for the compound FE of the shell with ring, and we use the matrix of the indices to formulate the equations of equilibrium for the entire reinforced shell. In matrix form, these equations appear as follows:

$$\boldsymbol{\Gamma}(\mathbf{U})\Delta\mathbf{U} = \mathbf{Q} - \mathbf{G}(\mathbf{U}). \tag{3}$$

Here, $\mathbf{G}(\mathbf{U})$ and $\boldsymbol{\Gamma}(\mathbf{U})$ are the gradient and the nonlinear Hessian matrix (matrix of second derivatives with respect to the generalized nodal displacements) of the potential strain

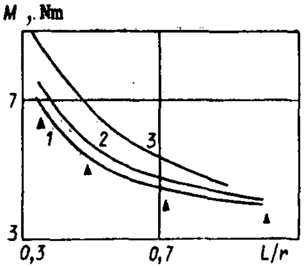

Fig. 3

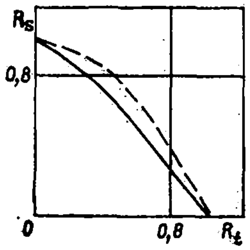

Fig. 4

energy, determined for a certain approximation **U** to the actual solution; $\Delta\mathbf{U}$ is a correction for **U**; **Q** is the vector of the nodal load.

Equation (3) is solved by the Newton—Kantorovich iteration method using the formulas

$$\Gamma(\mathbf{U}^n)\Delta\mathbf{U}^n = \mathbf{Q} - \mathbf{G}(\mathbf{U}^n),\ \mathbf{U}^{n+1} = \mathbf{U}^n + \Delta\mathbf{U}^n \tag{4}$$

(n is the number of the iteration). The iteration is stopped after satisfaction of the condition $||\Delta\mathbf{U}^n|| < \delta\ (\|\Delta\mathbf{U}^n\| = \max\limits_{i=1,\ldots,N} |\{\Delta\mathbf{U}^n\}_i|$, N is the order of system (3)).

In each iteration, Eq. (4), linear with respect to $\Delta\mathbf{U}^n$, is solved by the method of $\mathbf{L}^T\mathbf{AL}$-expansion of the matrix Γ (**L** is a triangular matrix and **A** is a diagonal matrix).

Stability is investigated on the basis of an energy criterion, confirming that a necessary and sufficient condition for stability of the equilibrium state of the deformed system is that the second variation of its potential derivative be positive. It follows from this that the matrix Γ should be positive-definite. Otherwise, the equilibrium will be unstable. The positive-definiteness of the matrix Γ can be checked by means of Sylvester's criterion, which in the present case reduces to checking the positiveness of the diagonal coefficients of matrix **A**. The transition of some coefficient to the negative region signifies a transition of the shell from the stable to the unstable equilibrium state.

To test the algorithm, we will examine a shell of the length L, radius r = 100 mm, and thickness h = 0.25 mm made of polystyrene with $E = 5.55 \cdot 10^3$ MPa and $\nu = 0.3$. Turning moments M_t are applied to the ends of the shell, which have been reinforced with rigid rings.

Figure 3 shows the dependence (curve 1) of the critical values of the moment M_t on the ratio L/r. Loss of stability occurs with the formation of from 22 (L/r = 0.3) to 15 (L/r = 1.0) oblique waves. Here, the triangles denote experimental results [4]. Also shown are the solutions obtained for the stability problem when it is assumed that the initial state is linear: curve 2 is the solution with allowance for the moments of the initial stress—

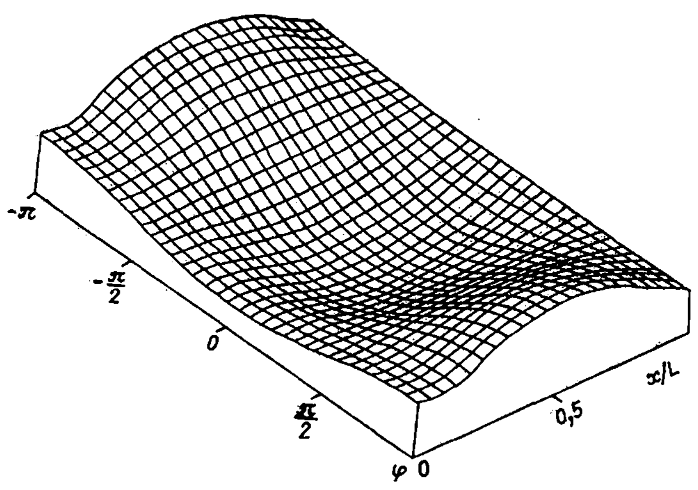

Fig. 5

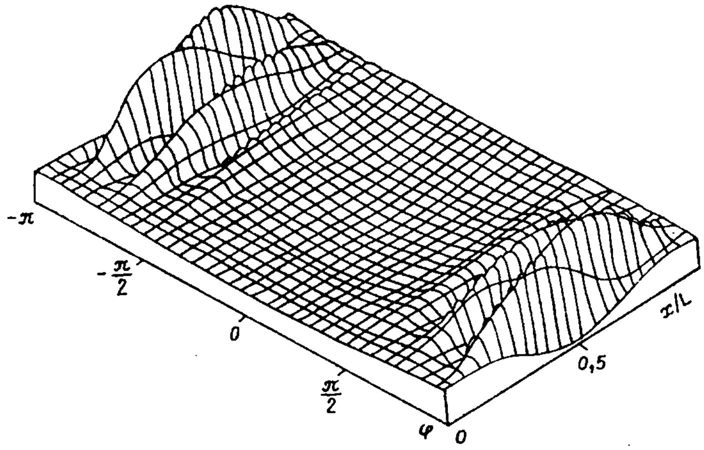

Fig. 6

strain state; curve 3 is the solution without allowance for these moments. The latter was obtained on the basis of the Donnel formula [1] for the critical shear stress

$$\tau_s = \frac{E}{1-\nu^2}\left(\frac{h}{L}\right)^2 \left\{4.6 + \left[7.8 + 1.67\left(\sqrt{1-\nu^2}\,\frac{L^2}{2Rh}\right)^{1.5}\right]^{0.5}\right\}.$$

It is evident that allowance for the moments of the initial state lowers the critical load up to 15%, while allowing for the moments and nonlinearity lowers the critical load up to 22%.

We will examine a cantilevered cylindrical shell (radius r, length L = 0.5r, thickness h = 0.01r) with a free end reinforced by a rigid ring. The shell is loaded by bending M_b and turning M_t moments applied to the free end. The following critical values were obtained for the moments in pure bending and pure torsion: $M_b^o = 1.175M_S$, $M_t^o = 1.536M_T$ ($M_S = \pi r^2 T_S$, $M_T = \pi r^2 h\tau_S$, $T_S = Eh^2/(r\sqrt{3(1-\nu^2)})$). To find the exact solution, it is sufficient to subdivide the shell lengthwise and about the circumference into m = 8 and k = 32 elements in pure bending and m = 10 and k = 20 elements in pure tension. The convergence of the solution in relation to the number of finite elements is shown in Table 1. In the case of combination loading with the moments M_b and M_t, we studied the stability of the shell with m = 12 and k = 34.

The solid curve in Fig. 4 shows the relation between the parameters $R_t = M_{t*}/M_t^o$ and $R_S = M_{b*}/M_b^o$ for combination loading (M_{t*} and M_{b*} are the critical values of the moments for combination loading). For comparison, the dashed line shows the analogous relation recommended in [1] for shells of moderate length.

Combination loading results in a complex configuration for both the initial deflection and the bifurcative deflection w. For $R_t = 0.84$, Fig. 5 shows the mode of deflection in the initial state. It is close to the configuration of the deflection w in pure bending. In the case of bifurcation, the deflections are localized in the region of axial compression (Fig. 6) and have the form of oblique waves with their maximum amplitude in the region of greatest compression.

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